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Commentary

Measures of diversity should include both matrix and vector inputs

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Hoffmann and Hoffmann (2008) commented in these pages about Jost's (2006) notions of "true" diversity. Hoffmann and Hoffman made several insightful points, such as it may be impossible to quantify diversity with a single number and there is a huge philosophical conundrum in designating something as the truth (see Gorelick, 2011a for further arguments regarding "true" diversity). Although not central to their arguments, Hoffmann and Hoffmann (2008) omitted a crucial feature of Jost's statistics: that entropy and its generalizations encompassed in the Hill–Tsallis indices often can and should map a matrix or higher-order array – and not just a vector – onto a scalar measure of diversity.

Marginal entropy maps a vector of probabilities $(p_1,p_2,\ldots,p_n)$ to a scalar and is given by $H=-\sum_{i=1}^{n} p_i \cdot log(p_i)$. Marginal entropy is a special case of the Hill–Tsallis indices $H^{(\alpha)}=\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{1-\alpha}}$ in the limit as α approaches 1, at least after taking $log(H^{(\alpha)})$ (Hill, 1973; Tsallis, 1988). The prefix 'marginal' as applied to entropy can be confusing in economic contexts. In probability and statistics, 'marginal' refers to a projection onto one or more lower dimensions of a joint (multi-dimensional) distribution. Shannon adopted this connotation, probably never envisioning that marginal entropy would be used in economics, resulting in a linguistic conflict with the calculus-derived economics connotation of 'marginal' as a derivative.

A potential problem is that diversity often has multiple orthogonal inputs and therefore should be quantified by mapping a matrix or higher-order array onto a scalar (see next paragraph and Gorelick and Bertram, 2010 for details). For example, we are often not just interested in abundance of species, but also these abundances in different age classes of individuals or these abundances over different geographic regions, such as counties or countries. Another example of β-diversity is encapsulating the information in a Leontief input–output matrix as a single number, such as measuring the robustness of the global carbon cycle in terms of the matrix of carbon fluxes. The concept of division of labor is equivalent to β-diversity and therefore sometimes also begs for matrix inputs (Gorelick, 2006; Gorelick and Bertram, 2007, 2010). Similar to β-diversity and division of labor, inequality indices are important in modern economics (Cowell, 2011), as exemplified by the following facetious bragging rights stemming from the 'Occupy' movement, "My Gini coefficient is bigger than your Gini coefficient" (Caplan, 2012).

One way to quantify β-diversity is take the ratio of mutual to marginal entropy, not just compute marginal entropy (Gorelick and Bertram, 2010; Gorelick et al., 2004). Mutual entropy maps a matrix $[p_{ij}]$ or higher order array $[p_{ijk}]$ onto a scalar via $I=\sum_{i,j,k} p_{ijk} \cdot log\left(\frac{p_{ijk}}{p_i \cdot p_j \cdot p_k}\right)$ (Cazelles, 2004; Gorelick, 2006; Gorelick and Bertram, 2010). While this has seemingly not been recognized in ecological economics, ecologists have noted this distinction between mutual and marginal entropy for over four decades (Colwell and Futuyma, 1971). One of Shannon's (1948) key innovations was mutual entropy. Marginal entropy, which is inappropriately called Shannon's index, was invented by Boltzmann (1872).

There are many reasonable definitions of β-diversity and the other similar notions of division of labor, segregation, and inequality (Gorelick and Bertram, 2007, 2010, and references therein), with virtually no consensus as to which one is best – if there is a universally best definition – because best definitions invariably depend upon context (Gorelick, 2011a,b; Wagner, 2010). Jost has done some beautiful work on β-diversity, using Hill–Tsallis indices, showing how overall diversity (γ-diversity) can be decomposed into unique orthogonal components of α- and β-diversity (Chao et al., 2010; Jost, 2007). Alternatively, social scientists have quantified β-diversity using the ratio of the square of coefficient of variation, which has matrix inputs, divided by its lower dimensional analog, variance, which has vector inputs (Gorelick and Bertram, 2007, 2010). The details do not matter (but see Gorelick and Bertram, 2010 for the details) so much as the realization that to measure diversity, segregation, or division labor we need (1) functions that have matrix inputs to account for the interactions between rows and columns and (2) functions that have vector inputs in order to normalize the results that arose from matrix inputs. This is why approaches using entropy need both mutual and marginal entropy and why approaches using variance need both coefficient of variation and its projection of variance or standard deviation.

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The paper that Hoffmann and Hoffmann (2008) critiqued, Jost (2006), broaches that β-diversity can be measured using conditional entropy, which is effectively equivalent to using mutual entropy, albeit without explicitly invoking the matrix structure of the input. But this does not provide any conceptual simplification because the input of conditional entropy is also a matrix because the vector $(p_1,p_2,\ldots,p_n)$ can itself be composed of elements p_i each of which is a vector of identical length, making $(p_1,p_2,\ldots,p_n)$ a matrix. For a matrix, mutual entropy equals marginal entropy minus conditional entropy.

For matrices, mutual entropy resembles a distance metric between rows and columns. Mutual entropy is a special case of Kullback–Leibler divergence (also known as 'relative entropy'), which is technically not a distance, and hence not so useful, because Kullback–Leibler divergence between rows and columns does not equal the divergence between columns and rows (Kullback and Leibler, 1951). Heuristically, given a specific row (or column) of a matrix, mutual information tells you whether a given column (or row) is more or less likely. And, unlike with Kullback–Leibler divergence, mutual entropy is independent of whether we first focus on rows or columns. Other measures of diversity, such as those derived from coefficient of variation do the same, tell you how much information you can glean about a column (or row) given a specific row (or column) (Gorelick and Bertram, 2010).

Although Shannon (1948) defined marginal and mutual entropy using logarithms with a base of two because he was dealing with two states, 0 and 1, choice of base of logarithms is largely irrelevant for discussions of diversity for two reasons. First, change of base amounts to multiplying entropy by a constant. So long as one is consistent with their choice of logarithmic base, comparisons of diversity can be made. Second, when measuring diversity, we invariably take ratios to normalize the results, such as ratio of mutual to marginal entropy. (Gorelick, 2006; Gorelick and Bertram, 2010; Jost, 2006), thereby rendering the ratio insensitive to choice of base.

My objective is not to malign the Hoffmanns nor Jost. Indeed, Jost (2007) has independently emphasized the importance of insuring that α- and β-diversity are independent and broached conditional entropy as a measure of β-diversity (Jost, 2006), while Hoffmann and Hoffmann (2008) raised important philosophical points, such as lack of a "true" measure of diversity. Rather my objective is to suggest that ecological economists sometimes invoke mutual entropy and entropies of high-order arrays – not just marginal entropy – and/or squares of coefficient of variation to measure diversity, division of labor, segregation, and information encapsulated in Leontief input–output models by quantifying interactions between rows and columns of an array. We need to look at statistics of entire matrices or higher-order arrays and all of their projections ('marginals') in order to quantify diversity.

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